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New results from glueball superpotentials and matrix models: the Leigh-Strassler deformation

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Abstract: Using the result of a matrix model computation of the exact glueball superpotential, we investigate the relevant mass perturbations of the Leigh-Strassler marginal "q" deformation of $\mathcal{N} = 4$ supersymmetric gauge theory. We recall a conjecture for the elliptic superpotential that describes the theory compactified on a circle and identify this superpotential as one of the hamiltonians of the elliptic Ruijsenaars-Schneider integrable system. In the limit that the Leigh-Strassler deformation is turned off, the integrable system reduces to the elliptic Calogero-Moser system which describes the $\mathcal{N} = 1^*$ theory. Based on these results, we identify the Coulomb branch of the partially mass-deformed Leigh-Strassler theory as the spectral curve of the Ruijsenaars-Schneider system. We also show how the Leigh-Strassler deformation may be obtained by suitably modifying Witten's M theory brane construction of $\mathcal{N} = 2$ theories.

Keywords: Supersymmetry and Duality, Supersymmetric Effective Theories, Matrix Models, Integrable Field Theories.

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1. Introduction

The motivation of the present paper is to derive some new results in supersymmetric gauge theories from the remarkable developments which relate glueball superpotentials of $\mathcal{N} = 1$ theories to matrix models [1]–[3].

Leigh and Strassler [4] discovered that $\mathcal{N} = 4$ gauge theory has two complex marginal deformations. One of them involves replacing the tree-level superpotential, which involves the commutator term, by the "q deformation":

$$W_{\rm cl} = i \,\mathrm{Tr}\ \Phi[\Phi^+, \Phi^-] \longrightarrow i\lambda \,\mathrm{Tr}\ \Phi[\Phi^+, \Phi^-]_\beta \,, \tag{1.1}$$

where we have defined the q-commutator

$$[\Phi^+, \Phi^-]_\beta \equiv \Phi^+\Phi^- e^{i\beta/2} - \Phi^-\Phi^+ e^{-i\beta/2} \,. \tag{1.2}$$

In the above, Φ and $\Phi^\pm$ are SU(N) adjoint-valued chiral fields (unlike [5] we will only consider the SU(N) theory and so we will drop the hats from SU(N)-valued fields). The resulting theory is known to be finite on some 2-complex dimensional surface in the space spanned by the two new couplings λ and β along with the complex gauge coupling τ. Away from the $\mathcal{N} = 4$ line, $\lambda = 1$, $\beta = 0$ with τ arbitrary, the theory only has $\mathcal{N} = 1$ supersymmetry.

In [5], generalizing the analysis of the $\mathcal{N} = 1^*$ theory in [3, 6, 7], we analysed certain relevant deformations of this space of theories by using the matrix formalism developed by Dijkgraaf and Vafa for calculating the exact glueball superpotentials of $\mathcal{N} = 1$ theories [1]–[3]. The perturbed theory is described by the tree-level superpotential

$$W_{\rm cl} = \mathrm{Tr}\ \left(i\lambda\Phi[\Phi^+, \Phi^-]_\beta + m\Phi^+\Phi^- + \mu\Phi^2\right) \,. \tag{1.3}$$

More generally one can replace the last term by an arbitrary function $W(\Phi)$. Rather remarkably in the confining vacuum the one-cut saddle-point solution of the matrix model is solvable thanks to [8]. In [5], we showed how the matrix model could also be solved around certain multi-cut solutions that describe all the massive vacua of the theory. Explicit formulae will be given later; however, on the basis of these explicit results we were able to conjecture an exact form for the "elliptic" superpotential of the theory. This is a superpotential induced by the final term in (1.3) when the theory is compactified to three dimensions. It is a rather useful quantity because, unlike the matrix model glueball superpotentials, it captures all the vacua in one go and the values of the condensates that are extracted from it are valid in the four-dimensional decompactification limit.

It is well known that the $\mathcal{N}=1^*$ and $\mathcal{N}=2^*$ theories are related to the elliptic Calogero-Moser system; in particular, the Coulomb branch of the $\mathcal{N}=2^*$ theory is the spectral curve of the integrable system and the elliptic superpotential that arises on breaking to $\mathcal{N}=1^*$ is one of the hamiltonians. In this paper, we ask whether the matrix model results of [5] can be used to deduce whether this relation to integrable systems is maintained under the Leigh-Strassler q deformation? The answer is yes, since it transpires that the Leigh-Strassler q deformation involves a natural one parameter deformation of the elliptic Calogero-Moser system known as the Ruijsenaars-Schneider system. This result allows us to solve the Seiberg-Witten theory of the Leigh-Strassler q deformation of the $\mathcal{N}=2^*$ theory. In other words, we find the exact description of this $\mathcal{N}=1$ Coulomb branch.

2. Review of the relevant deformations of $\mathcal{N}=4$

In this section we will review the story of the relevant deformations of $\mathcal{N}=4$. In other words, we take $\lambda=1$ and $\beta=0$ in (1.3). This theory is known as the $\mathcal{N}=1^*$ theory. The vacuum structure of this theory was originally determined by compactifying the theory on $\mathbb{R}^3\times S^1$ [9] (see also [10]). In order to describe how this works it is useful to think of the $\mathcal{N}=1^*$ deformation in two stages: firstly with only the mass m non-zero, which describes the so-called $\mathcal{N}=2^*$ theory, and then with the final mass μ turned on.

The $\mathcal{N}=2^*$ theory in four dimensions has a Coulomb branch which was described by Donagi and Witten [11]. In particular, it is the moduli space of the Seiberg-Witten curve which in this case is a certain N-fold ramified cover of the basic torus E_τ with complex structure τ the complex gauge coupling. In the following, we choose a normalization in which the periods of E_τ are:

$$2\omega_1=2\pi i\,,\qquad 2\omega_2=2\pi i\tau\,. \tag{2.1}$$

The Seiberg-Witten curve is also the spectral curve of the Calogero-Moser integrable system [13]–[15] describing the interaction N particles according to the hamiltonian [19, 20]

$$H=\sum_a p_a^2+m^2\sum_{a\neq b}\wp(x_a-x_b)\,, \tag{2.2}$$

with momenta p_a and positions x_a. In the following, we shall freeze the trivial centre-of-mass motion by choosing

$$\sum_a p_a=\sum_a x_a=0\,. \tag{2.3}$$

This amounts to restricting to the $\mathrm{SU}(N)$ gauge group without an additional $\mathrm{U}(1)$ factor. In the above, $\wp(x)$ is the Weierstrass function defined on the torus E_τ.

Integrability is manifest in the Lax formalism. One defines the Lax matrix with elements

$$\phi_{ab}(z) = p_a\delta_{ab} + im(1-\delta_{ab})\frac{\sigma(x_{ab}+z)}{\sigma(x_{ab})\sigma(z)}\,, \tag{2.4}$$

where $\sigma(z)$ is the Weierstrass sigma function and $x_{ab} = x_a - x_b$. The time evolution can then be written via another matrix M:

$$\dot{\phi} = [M, \phi] \tag{2.5}$$

which clearly leaves invariant the spectrum of $\phi(z)$. In particular, the spectral curve Σ

$$F(v,z) = \det\big(v1_{[N]\times[N]} - \phi(z)\big) = 0 \tag{2.6}$$

is left invariant. This curve is an N-fold branched covering of the basic torus E_τ:

$$F(v, z+2\omega_1) = F(v, z+2\omega_2) = F(v,z) \tag{2.7}$$

and plays the rôle of the Seiberg-Witten curve for the theory. A basis for the space of $N-1$ hamiltonians is obtained by taking the finite parts of $\mathrm{Tr}\,\phi^k(z)$, $k = 2,\ldots,N$, around the poles at $z = 0$. In particular,

$$\mathrm{Tr}\ \phi^2(z) = -m^2N(N-1)\wp(z) + H\,, \tag{2.8}$$

where H is (2.2).

The Coulomb branch of the $\mathcal{N} = 2^*$ theory is identified with the moduli space of the spectral curve (2.6) of the complexified Calogero-Moser system — so x_a and p_a are taken to be complex — parameterized by the $N-1$, now complex, hamiltonians. The relation with the integrable system becomes even more satisfying once the theory is compactified on a circle: in that case the Coulomb branch doubles in dimension because there are $N-1$ Wilson lines and dual photons which can be amassed into $N-1$ additional complex scalar fields. These variables are naturally valued on a multi-dimensional torus which is nothing but the space of the angle variables of the Calogero-Moser system conjugate to the hamiltonian, or action, variables. The multi-dimensional torus is naturally identified with the jacobian variety $\mathfrak{J}(\Sigma)$ of the spectral curve. So in the compactified theory the Coulomb branch is naturally identified with the whole phase space of the integrable system.

Now we can turn on the final mass deformation μ. This lifts the Coulomb branch according to a superpotential which is precisely the quadratic hamiltonian H in (2.2). This is the exact elliptic superpotential of [9]. Actually more general deformations of the form $\mathrm{Tr}\ W(\Phi)$ can be considered as these simply correspond to some linear combination of the $N-1$ hamiltonians.

One way to understand why the superpotential is exact and to more fully elucidate the relation with the integrable system is to realize the whole set-up within string theory. We briefly describe the chain of arguments. One starts with Witten's elliptic brane construction in Type IIA string theory [16]. The background spacetime is $\mathbb{R}^9 \times S^1$. There are N D4-branes whose world-volume lies in $\mathbb{R}^4 \times S^1$ parameterized by x^n, $n = 0,1,2,3,6$, where

x^6 is the periodic coordinate. There is one NS5-brane with a world-volume along x^n, $n = 0, 1, 2, 3, 4, 5$. The low energy theory on the D4 branes is then four-dimensional $\mathcal{N} = 4$ supersymmetric gauge theory. Up until now, the single NS5 plays no rôle. However by twisting the spacetime we can break to $\mathcal{N} = 2^*$. This is achieved by a non-trivial fibration of the complex direction $v = x^4 + ix^5$ over the x^6 circle:

$$x^6 \longrightarrow x^6 + 2\pi L\,, \qquad v \longrightarrow v + m\,. \tag{2.9}$$

In this case the D4-branes have to split at the NS5-brane and the resulting theory at low energies is the $\mathcal{N} = 2^*$ theory. In order to include quantum corrections one now lifts the configuration to M theory [16]. A new dimension appears parameterized by x^{10} which, along with x^6, forms a torus E_τ with complex structure τ, the underlying complex gauge coupling of the theory. In M theory the configuration of D4-branes and NS5-brane lifts to a single M theory 5-brane with a world volume $\mathbb{R}^4 \times \Sigma$, where Σ is a 2-surface embedded non-trivially in the four-dimensional space $\mathbb{R}^2 \times E_\tau$, parameterized by the two complex coordinates (v, z). The embedding is described by the Seiberg-Witten curve and has the form

$$F(v, z) = v^N - f_1(z)v^{N-1} + f_2(z)v^{N-2} - \cdots + (-1)^N f_N(z) = 0\,. \tag{2.10}$$

The functions $f_a(z)$ are elliptic on the torus E_τ (which we take, as before, to have periods $(2\omega_1 = 2\pi i, 2\omega_2 = 2\pi i\tau)$), with the following analytic structure. The elliptic functions $f_a(z)$ have a pole of order a at $z = 0$ and upon a suitable shift in v, the singularities of $F(v, z)$ can be converted into a simple pole at $z = 0$. It is not difficult to show [11, 16] that the curve (2.10) is precisely the spectral curve of the Calogero-Moser system (2.6). Note that in order to describe SU(N), rather than U(N), we decouple the centre-of-mass motion and this sets $f_1(z) = 0$.

As an alternative to lifting to M theory, following Kapustin [17], we take a different route that leads to the same result but is more suitable for our needs. The idea is to compactify one of the spacetime directions of the D4-branes, say x^3, on a circle of radius R. For small radius R, we can now perform a T-duality in x^3 to yield the Type IIB configuration of D3-branes spanning x^0, x^1, x^3, x^6. Under this duality, the string coupling is transformed to $g_s' = \sqrt{\alpha'}/R$. We follow this with an S-duality on the four-dimensional theory on the D3-branes. Finally, we perform, once again, a T-duality in x^3 to return to a Type IIA configuration with D4-branes spanning x^0, x^1, x^2, x^3, x^6. However, due to the intervening S-duality, the radius of the x^3 is not returned to its original value. The new radius is $Rg_s' = g_s\sqrt{\alpha'}$. In other words, it is independent of the radius R.[1] The theory describing the collective dynamics of these D4-branes is the mirror dual, or "magnetic", theory. It is a five-dimensional theory compactified on $\mathbb{R}^3 \times T^2$. The most significant fact is that the torus T^2 has complex structure τ, the complex gauge coupling of the original theory, and so is identified with the basic torus E_τ but now realized in the (x^3, x^6) space rather than the (x^6, x^{10}) space of the M theory construction.

The discussion so far has been simplified because we have ignored the fact that there is an NS5-brane in the original Type IIA set-up on which the D4-branes can split when

[1]All memory of R is not lost because the string coupling in the dual theory is $R^2/(g_s\alpha')$.

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the mass m is non-vanishing. Under the first T-duality the NS5-brane become a Type IIB NS5-brane. Then under S-duality it becomes a D5'-brane (to distinguish it from the other D-branes in the problem). Finally the T-duality around x^3 changes it into an D4'-brane spanning x^0, x^1, x^2, x^4, x^5, but localized at points on the (x^3, x^6) torus. As usual in a mirror transform we have mapped the Coulomb branch of the original theory, where the D4-branes were prevented from moving off the NS5-branes, to the Higgs branch of the magnetic theory.

The configuration that we are considering preserves eight real supersymmetries. So we have a realization of the Coulomb branch of the 3-dimensional theory as the Higgs branch of an "impurity" gauge theory with eight real supercharges. This is why the mirror map is a useful device. The Higgs branch will not be subject to quantum corrections and in this way we are able to "solve" the theory. It is naturally described by a set of D- and F-flatness equations which involve the, suitably normalized, components of the dual SU(N) gauge field $\tilde{A}_{z,\bar{z}} = \frac{1}{2}(\tilde{A}_3 \pm i\tilde{A}_6)$ of the D4-branes along the torus[2] and the adjoint-valued complex scalar field ϕ describing the fluctuations of the D4-branes in the x^4, x^5 direction. In addition, the D4'-brane impurity gives rise to a hypermultiplet $(Q_a\, \tilde{Q}_a)$ transforming in the $(\boldsymbol{N}, \overline{\boldsymbol{N}})$-representation of SU($N$), which is *localized* at a point on the torus which we choose at $z = 0$. The D- and F-flatness conditions, respectively, with some convenient choice of normalization of the hypermultiplets, read [10, 17]

$$\Big(\tilde{F}_{z\bar{z}} - [\phi, \phi^\dagger]\Big)_{ab} = -i\pi\delta^2(z,\bar{z})(Q_a Q_b^\dagger - \tilde{Q}_a^\dagger \tilde{Q}_b)\,, \tag{2.11a}$$

$$\Big(\tilde{\mathcal{D}}_{\bar{z}}\phi\Big)_{ab} = i\pi\delta^2(z,\bar{z})(Q_a\tilde{Q}_b - m\delta_{ab})\,. \tag{2.11b}$$

Here, (2.11a) is a real equation and (2.11b) is a complex equation and $\tilde{\mathcal{D}}_{\bar{z}}\phi = \partial_{\bar{z}}\phi + [\tilde{A}_{\bar{z}}, \phi]$. Notice how the mass enters into the F-flatness equation. These equations are a generalization of Hitchin's self-duality equations reduced to two dimensions [18].

The Coulomb branch of the 3-dimensional theory is then the solution of the D- and F-flatness conditions modulo local SU(N) gauge transformations on the torus. The construction is an example of an infinite hyper-Kähler quotient and so the Coulomb branch of the 3-dimensional theory is a $4(N-1)$-dimensional hyper-Kähler space. As usual as long as we are interested in holomorphic quantities we can relax the D-flatness condition and then solve for the F-flatness condition moduli complex local gauge transformations — those valued in SL($N, \mathbb{C}$).

To proceed, it is very convenient to use up (most of) the local part of the quotient group, SL($N, \mathbb{C}$), to transform the anti-holomorphic component $A_{\bar{z}}$ into a constant diagonal matrix:

$$\tilde{A}_{\bar{z}} = \frac{\pi i}{2(\bar{\omega}_2\omega_1 - \bar{\omega}_1\omega_2)}\mathrm{diag}(x_1, \ldots, x_N) \tag{2.12}$$

with $\sum_{a=1}^N x_a = 0$. The only local transformations that remain act by shifting the x_a by periods of the torus:

$$x_a \to x_a + 2n\omega_1 + 2m\omega_2\,, \qquad m, n \in \mathbb{Z}\,. \tag{2.13}$$

[2]We perform an overall re-scaling of the torus T^2 so that it becomes precisely E_τ parameterized by the holomorphic coordinate z with periods $2\omega_1 = 2\pi i$ and $2\omega_2 = 2\pi i\tau$.

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The remaining global part of the gauge group is also fixed, up to permutations of the x_a, by choosing

$$\tilde{Q}_a = 1\,. \tag{2.14}$$

We can now solve explicitly for ϕ to get a very concrete parameterization of the Coulomb branch. With $\tilde{A}_{\bar{z}}$ diagonal, the diagonal elements ϕ_{aa} are meromorphic functions on the torus with a possible simple pole at $z = 0$. However, there are no such functions other than a constant; consequently,

$$\phi_{aa} = p_a \qquad \text{and} \qquad Q_a = m\,. \tag{2.15}$$

where the p_a with $\sum_{a=1}^{N} p_a = 0$ are new parameters.

The off-diagonal elements are

$$\phi_{ab}(z,\bar{z}) = im\frac{\sigma(x_{ab}+z)}{\sigma(x_{ab})\sigma(z)}e^{\psi(z,\bar{z})x_{ab}} \qquad (a \neq b)\,. \tag{2.16}$$

Here $x_{ab} \equiv x_a - x_b$, and we have defined

$$\psi(z,\bar{z}) = \frac{1}{\bar{\omega}_2\omega_1 - \bar{\omega}_1\omega_2}\left[\zeta(\omega_2)(\bar{\omega}_1 z - \omega_1\bar{z}) - \zeta(\omega_1)(\bar{\omega}_2 z - \omega_2\bar{z})\right]. \tag{2.17}$$

One can readily verify that $\phi_{ab}(z,\bar{z})$ is periodic on the torus. Furthermore, a shift of x_a by a lattice vector $2\omega_\ell$, can be undone by a large gauge transformation on the torus as anticipated earlier. Up to a simple diagonal gauge transformation,

$$U_{ab} = e^{-\psi(z,\bar{z})x_a}\delta_{ab}\,, \tag{2.18}$$

the matrix ϕ, with elements (2.15) and (2.16), is equal to the Lax matrix of the elliptic Calogero-Moser system (2.4) where the x_a are the positions and the p_a are the momenta.

We now have an explicit parameterization of the 3-dimensional Coulomb branch furnished by $\{p_a, x_a\}$ with

$$\sum_a p_a = \sum_a x_a = 0\,. \tag{2.19}$$

As we have already alluded to above, there is also a completely integrable dynamical system for which x_a are the positions and p_a are momenta with the usual Poisson bracket structure. It is the elliptic Calogero-Moser system [19, 20]. In particular, as we have already stated, the spectral curve (2.6) is precisely the Seiberg-Witten curve Σ of the four-dimensional theory before compactification to three dimensions. Since the dynamical system is completely integrable, there are $N-1$ (complex) hamiltonians. These are identified with coordinates on the Coulomb branch of the four-dimensional theory. The conjugate angle variables — also complex — take values in the jacobian of Σ.

A basis in the space of the Poisson-commuting hamiltonians of the dynamical system is obtained by taking the gauge invariant quantities $\mathrm{Tr}\phi^k(z)$, $k = 2, \ldots, N$, for some fixed z. These hamiltonians parameterize the Coulomb branch of the four-dimensional theory. In particular, the quadratic hamiltonian is the finite part of

$$\mathrm{Tr}\ \phi(z)^2 = -N(N-1)m^2\wp(z) + \sum_{a=1}^{N} p_a^2 + m^2\sum_{a\neq b}\wp(x_{ab}) \tag{2.20}$$

around $z = 0$.

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Now that we have established the description of the Coulomb branch of the compactified theory in terms of the Calogero-Moser integrable system, we can break to $\mathcal{N}=1^*$ by adding the general perturbation

$$\sum_{k=2}^{N}\mu_k \mathrm{Tr}\,\Phi^k \tag{2.21}$$

to the tree-level superpotential. In the Higgs branch description of the compactified theory this gives rise to an exact "elliptic" superpotential which is some linear combination of hamiltonians:

$$W_{\rm eff}(x_a,p_a)=\sum_{k=2}^{N}\mu_k H_k\,. \tag{2.22}$$

This superpotential lifts the three-dimensional Coulomb branch and importantly is independent of the compactification radius and so is equally valid in the four-dimensional limit. The only subtlety involved is in identifying the coordinates $\{H_k\}$ in the space of hamiltonians. This problem involves resolving operator mixing ambiguities [9, 10, 21]. For the quadratic perturbation $\mu\mathrm{Tr}\,\Phi^2$ the situation is simple and the exact elliptic superpotential is

$$\frac{1}{\mu}W_{\rm eff}(x_a,p_a)=\sum_{a=1}^{N}p_a^2+m^2\sum_{a\neq b}\wp(x_{ab}) \tag{2.23}$$

up to an additive constant which is not physically significant.

We can go on to consider the vacuum structure of the theory by extremizing $W_{\rm eff}(x_a, p_a)$. First of all, it is clear that in any vacuum $p_a=0$. There are two classes of vacua: the massive and massless. The former have been completely determined [9] while the classification of the latter is still an unsolved problem [21]. The massive vacua have a very beautiful interpretation from the point-of-view of the dynamical system [6, 10]: they are precisely *equilibrium configurations* with respect to the space of flows defined by the $N-1$ hamiltonians.[3] The point is that the massive vacua correspond to points of the four-dimensional Coulomb branch for which Σ degenerates to a torus: cycles pinch off and one is left with an N-fold un-branched cover of the basic torus E_τ. This means that the jacobian Variety $\mathfrak{J}(\Sigma)$ itself degenerates: at these points the period matrix only has rank 1, with non-zero eigenvalue τ. The remaining torus is associated with the centre-of-mass motion of the integrable system, so the overall U(1) factor, which we have removed by (2.19). So at a massive vacuum, the remaining angle variables must stay fixed under any time evolution. Since the hamiltonians are by definition constants of the motion, this means that the entire dynamical system must be static at a massive vacuum and the system is at an equilibrium point. Consequently, a massive vacuum is not only a critical point of the quadratic hamiltonian but simultaneously of all the other $N-2$ hamiltonians.

The simplest kind of massive vacua are labelled by two integers p and q with $pq=N$. All the other cases can be generated from these by modular transformations of τ (in fact all

[3] Here, "time" is an auxiliary concept referring to evolution in the dynamical system and not a spacetime concept in the field theories under consideration.

the massive vacua lie on a single orbit of the modular group). The critical point is then [9]

$$x_a \in \left\{ \frac{2r}{q}\omega_1 + \frac{2s}{p}\omega_2, \quad 0 \le r < q,\ 0 \le s < p \right\}. \tag{2.24}$$

The proof that this is a critical point of W_{eff} is delightfully simple. One only needs to use the fact that $\wp'(z)$ is an odd elliptic function. Terms in the sum $\sum_{b(\neq a)} \wp'(x_{ab})$ either cancel in pairs or vanish because x_{ab} is a half-lattice point. As we mentioned, the set (2.24) does not exhaust the set of massive vacua. For a given pair (q,p) we can generate $q-1$ additional vacua by replacing $\tau \to \tau + l/p$, $l = 0, \ldots, q-1$. So the total number of massive vacua is equal to $\sum_{p|N} p$, as expected on the basis of a semi-classical analysis [9, 11].

In the vacua (2.24) one can write down an expression for the superpotential in terms of the 2^{nd} Eisenstein series:

$$W_{\text{eff}} = -\frac{\mu m^2 N p^2}{12} E_2\left(p^2 \frac{\tau}{N}\right), \tag{2.25}$$

up to a vacuum-independent constant.

3. The Leigh-Strassler deformation

Now we consider the deformed theory. We start with the expressions that generalize the superpotential in a subset of the massive vacua (2.25) that we derived from the matrix model formalism [5]:

$$W_{\text{eff}} = \frac{pN\mu M^2}{2\lambda^2 \sin\beta} \cdot \frac{\theta_1'(p\beta/2|p^2\hat{\tau}/N)}{\theta_1(p\beta/2|p^2\hat{\tau}/N)} - \frac{N\mu M^2}{4\lambda^2 \sin^2(\beta/2)}. \tag{3.1}$$

In this expression, we have defined the renormalized gauge coupling

$$\hat{\tau} = \tau - \frac{iN}{\pi} \ln\lambda. \tag{3.2}$$

In the limit $\beta \to 0$ and $\lambda \to 1$ (3.1) reduces to (2.25).

Using (3.1) we will attempt to reverse the route followed in the last section. First of all, in [5] we identified the following elliptic superpotential for which the configurations (2.24) are still critical points and for which the superpotential takes the values (3.1) up to an additive vacuum-independent constant. The relevant expression is

$$W_{\text{eff}}(x_a) = \frac{im^2\mu}{2\lambda^2 \sin\beta} \sum_{a \neq b} \left(\zeta(x_a - x_b + i\beta) - \zeta(x_a - x_b - i\beta) \right), \tag{3.3}$$

where $\zeta(z)$ is the Weierstrass zeta-function which can be defined via $\wp(z) = -\zeta'(z)$ where $\wp(z)$ is the Weierstrass function for the torus $E_{\hat{\tau}}$ with periods $2\omega_1 = 2\pi i$ and $2\omega_2 = 2\pi i\hat{\tau}$. (Definitions of the elliptic functions that arise can be found in standard texts, for example [12].)

The question is whether $W_{\text{eff}}(x_a)$ can be identified with a hamiltonian of a known integrable system? The answer is yes, when one investigates the known integrable systems one finds a natural candidate which is a one parameter deformation of the Calogero-Moser system. It is known as the Ruijsenaars-Schneider system [22]–[24] or sometimes known as the relativistic elliptic Calogero-Moser system. The first and second hamiltonians can be written

$$H_1 = \sum_a \rho_a \,, \qquad H_2 = \sum_{a \neq b} \rho_a \rho_b \frac{1}{\wp(i\beta) - \wp(x_{ab})} \,. \tag{3.4}$$

The ρ_a are not directly the momenta conjugate to x_a, in fact these are p_a defined via

$$e^{p_a} = \rho_a \prod_{b(\neq a)} \frac{1}{\sqrt{\wp(x_{ab}) - \wp(i\beta)}} \,. \tag{3.5}$$

In order to relate this system to our superpotential (3.1), we will impose the constraint

$$\sum_a \rho_a = N \left(\frac{im^2 \wp'(i\beta)}{2\lambda^2 \sin\beta} \right)^{1/2} \,, \tag{3.6}$$

which along with $\sum_a x_a = 0$, is the analogue of freezing out the centre-of-mass motion that we did in the $\mathcal{N} = 1^*$ in order to describe the $SU(N)$, rather then $U(N)$, theory. Notice that this constraint is *not* the same as freezing out the centre-of-mass motion of the Ruijsenaars-Schneider system since this would be $\sum_a p_a = 0$. However, we will shortly show how this constraint arises naturally.

It is then straightforward to show that $H_2(x_a, \rho_b)$, subject to the constraint (3.6), is equal to (3.1), up to an additive vacuum-independent constant, once the ρ_a are integrated out. This procedure gives

$$\rho_a = \left(\frac{im^2 \wp'(i\beta)}{2\lambda^2 \sin\beta} \right)^{1/2} \,. \tag{3.7}$$

In order to complete the equality with (3.1) one uses the elliptic function identity

$$\frac{\wp'(i\beta)}{\wp(i\beta) - \wp(z)} = \zeta(z + i\beta) - \zeta(z - i\beta) - 2\zeta(i\beta) \,. \tag{3.8}$$

By writing

$$\rho_a = \left(\frac{i\wp'(i\beta)}{2\lambda^2 \sin\beta} \right)^{1/2} (m + i\tilde{p}_a \sigma(i\beta)) \,, \tag{3.9}$$

for new coordinates $\tilde{p}_a$, with $\sum_a \tilde{p}_a = 0$, and then taking the limit $\beta \to 0$ (and $\lambda \to 1$), the 2^{nd} hamiltonian H_2 reduces to the quadratic hamiltonian of the elliptic Calogero-Moser with momenta $\tilde{p}_a$. We remark that this is apparently different from the usual limit of the Ruijsenaars-Schneider system that gives the elliptic Calogero-Moser system. In that limit, one takes $p_a \sim \mathcal{O}(\beta)$ and it is the first hamiltonian H_1 that gives the quadratic hamiltonian of the elliptic Calogero-Moser system.

It is clear that these facts identify the Coulomb branch of the 3-dimemsional $SU(N)$ theory, *i.e.* the theory with tree-level superpotential

$$W_{\text{cl}} = \text{Tr}\, \left(i\lambda\Phi[\Phi^+, \Phi^-]_\beta + m\Phi^+\Phi^- \right) \,, \tag{3.10}$$

as the spectral curve of the Ruijsenaars-Schneider system. The latter can be constructed by a simple deformation of the Hitchin system description of the elliptic Calogero-Moser system described in the last section. The idea is to demand that the Lax matrix $\phi(z,\bar{z})$ is no longer periodic on the underlying torus $E_{\hat{\tau}}$ (with complex structure $\hat{\tau}$) rather there is a non-trivial boundary condition:

$$\phi(z+2\omega_1)=\phi(z)\,,\qquad \phi(z+2\omega_2)=e^{i\beta}\phi(z)\,. \tag{3.11}$$

We can solve the F-flatness condition (2.11b) modulo complex gauge transformation as before but now incorporating the boundary condition (3.11). Choosing the gauge (2.12) and (2.14), as before, the solution for the elements of the Lax matrix is

$$\phi_{ab}(z,\bar{z})=i(Q_a-m\delta_{ab})\frac{\sigma(x_{ab}-i\beta+z)}{\sigma(x_{ab}-i\beta)\sigma(z)}e^{\psi(z,\bar{z})x_{ab}+i\beta\zeta(\omega_1)z/\omega_1}\,, \tag{3.12}$$

with the tracelessness constraint, since we work in SU(N) rather than U(N),

$$\sum_{a=1}^{N}Q_a=Nm\,. \tag{3.13}$$

Notice that in contrast to the $\mathcal{N}=1^*$ case, the diagonal elements are not constant and Q_a is not forced to be m: in fact the Q_a will be related to the conjugate momenta.

The quadratic hamiltonian is extracted from the gauge invariant quantity

$$\begin{aligned}\mathrm{Tr}\ \phi^2(z) &= -e^{2i\beta\zeta(\omega_1)z/\omega_1}\frac{\sigma(z-i\beta)^2}{\sigma(z)^2\sigma(i\beta)^2}\sum_{ab}(Q_a-m\delta_{ab})(Q_b-m\delta_{ab})\frac{\wp(x_{ab})-\wp(z-i\beta)}{\wp(x_{ab})-\wp(i\beta)}\\ &= e^{2i\beta\zeta(\omega_1)z/\omega_1}\frac{\sigma(z-i\beta)^2}{\sigma(z)^2\sigma(i\beta)^2}(\wp(i\beta)-\wp(z-i\beta))\times\\ &\quad\times\left\{-\frac{N(N-1)m^2}{\wp(i\beta)-\wp(z-i\beta)}+\sum_{a\neq b}Q_aQ_b\frac{1}{\wp(i\beta)-\wp(x_{ab})}\right\}.\end{aligned} \tag{3.14}$$

We now recognize the second term in braces as proportional to the hamiltonian H_2 in (3.4) with

$$\rho_a=\left(\frac{i\wp'(i\beta)}{2\lambda^2\sin\beta}\right)^{1/2}Q_a \tag{3.15}$$

and the (3.13) is the traceless condition (3.6). It is not so obvious that the Lax matrix (3.12) is equivalent to the more conventional form in [24]. Firstly, we perform the diagonal gauge transformation (2.18). Then we notice that the part involving m is a constant proportional to the identity matrix and so we can shift this away without affecting the spectral curve or dynamics. Next, we have to multiply by the overall factor

$$i\frac{\sigma(i\beta)\sigma(z)}{\sigma(z-i\beta)}e^{-i\beta\zeta(\omega_1)z/\omega_1} \tag{3.16}$$

and, finally, shift the spectral parameter $z\to z+i\beta$. The resulting Lax matrix is then the one quoted in [24]:

$$\phi_{ab}=Q_a\frac{\sigma(x_{ab}-z)}{\sigma(x_{ab}-i\beta)}\,. \tag{3.17}$$

We remark that our construction of the Lax matrix via a simple modification of the Hitchin system of the elliptic Calogero-Moser system would appear to be simpler than the alternative constructions in [25, 26].

So we have demonstrated that the elliptic superpotential that described the Leigh-Strassler deformed $\mathcal{N} = 1^*$ is the quadratic hamiltonian of the Ruijsenaars-Schneider system. Furthermore, it now follows that the Coulomb branch of the four-dimensional Leigh-Strassler deformed $\mathcal{N} = 2^*$ theory with tree-level superpotential

$$W_{\rm cl} = {\rm Tr}\,\left(i\lambda\Phi[\Phi^+,\Phi^-]_\beta + m\Phi^+\Phi^-\right) \tag{3.18}$$

is described by the spectral curve of the Ruijsenaars-Schneider system, *i.e.* as in (2.6) but with the deformed Lax matrix (3.12). We can write the curve as in (2.10):

$$F(v,z) = v^N - f_1(z)v^{N-1} + f_2(z)v^{N-2} - \cdots + (-1)^N f_N(z) = 0\,, \tag{3.19}$$

but with modified conditions of the $f_a(z)$. These functions have the same pole structure as before; however they are no longer elliptic functions rather they incorporate the non-trivial boundary conditions on $E_{\hat\tau}$:

$$f_a(z+2\omega_1) = f_a(z)\,, \qquad f_a(z+2\omega_2) = e^{ia\beta}f_a(z)\,, \tag{3.20}$$

so that

$$F(v,z+2\omega_1) = F(v,z)\,, \qquad F(v,z+2\omega_2) = F(ve^{i\beta},z)\,. \tag{3.21}$$

As before in the SU(N) theory $f_1(z) = 0$. As an example, in the SU(2) theory with a suitable re-scaling of v,

$$v \longrightarrow e^{2i\beta\zeta(\omega_1)z/\omega_1}\frac{\sigma(z-i\beta)^2}{\sigma(z)^2\sigma(i\beta)^2}v\,, \tag{3.22}$$

the spectral curve is

$$v^2 + m^2 + Q(2m-Q)\frac{\wp(i\beta)-\wp(z-i\beta)}{\wp(x)-\wp(i\beta)} = 0\,, \tag{3.23}$$

where $x \equiv x_{12}$ and $Q = Q_1 = 2m - Q_2$.

From this result it is now possible to identify how one must modify Witten's brane construction of the $\mathcal{N} = 2^*$ theory [16] in order to incorporate the Leigh-Strassler q deformation. As one goes around the x^6 circle one needs to incorporate a non-trivial rotation $v \to e^{i\beta}v$. However, it is not so obvious how one can decouple the overall U(1) factor in the brane set-up.

As a final comment, it is intriguing that the Ruijsenaars-Schneider system also plays a rôle in the $\mathcal{N} = 2^*$ theory lifted to five dimensions and then compactified on a circle of radius R [27]. In this case, the deformation parameter β is identified with Rm. However, apparently the resulting spectral curve is different because in the five-dimensional case we have the constraint $\sum_a p_a = 0$, where the momenta are defined in (3.5), rather than the constraint (3.6) in the Leigh-Strassler case. For example, in the case of SU(2) the resulting spectral curve can be written as

$$v^2 + 2i\cosh p\sqrt{\wp(x)-\wp(iRm)}\,v + \wp(iRm) - \wp(z) = 0\,, \tag{3.24}$$

to compare with (3.23).

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